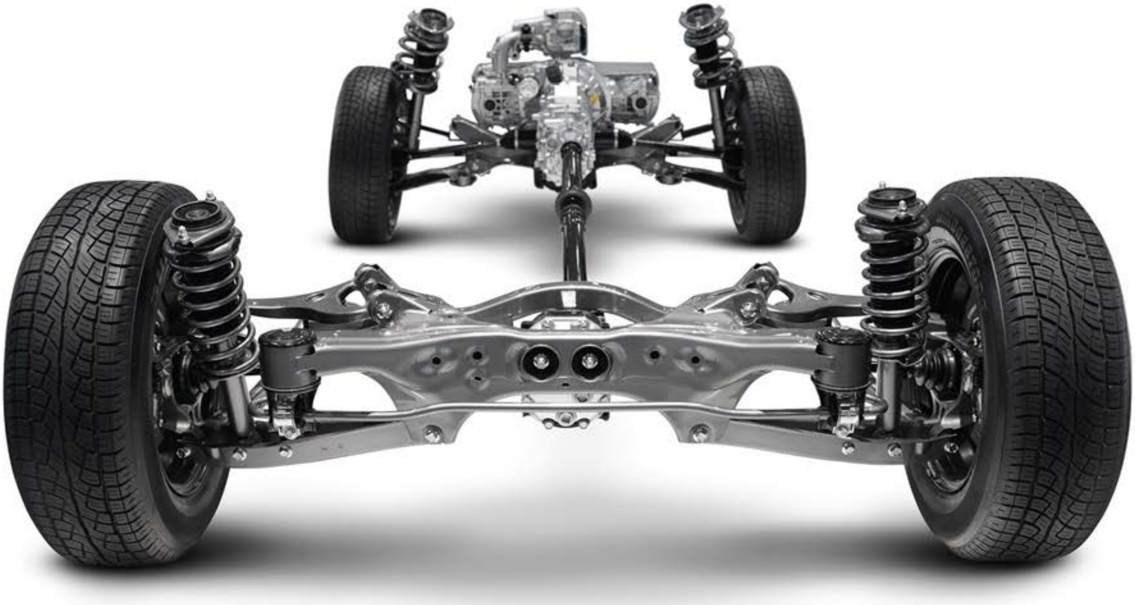
# MECHANISMS AND MACHINES

## KINEMATICS, DYNAMICS, AND SYNTHESIS



MICHAEL M. STANIŠIĆ

# Mechanisms and Machines:
## Kinematics, Dynamics, and Synthesis

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*To **L**auren, **E**mily, and **O**livia.*

# Contents

# Preface

This text is an introduction to the analysis and synthesis of mechanisms and machines, with an emphasis on the first. The intended audience is undergraduates who are studying mechanical engineering but the audience could also include students enrolled in multidisciplinary programs such as mechatronics or biomechanics.

## The Vector Loop Method and Kinematic Coefficients

My main motivation for writing this book is to introduce the vector loop method and kinematic coefficients into an introductory Mechanisms and Machines course and to present this subject matter entirely in terms of them. Over the years, I have found that using this method provides students with systematic and easily computerized procedures for determining the kinematic and dynamic properties of machines and mechanism.

According to Hall[1], kinematic coefficients were first introduced by Eksergian[2] who formulated the kinematic and dynamic equations of motion of a machine in terms of "velocity ratios and their derivatives with respect to the fundamental coordinate of the mechanism . . ."

These velocity ratios and their derivatives are the kinematic coefficients. I suspect the method did not catch on then due to its computational intensity. At that time, digital computing was non-existent and analytical results were primarily obtained graphically. The legacy of that continues as most texts today spend considerable time presenting graphical methods, such as velocity and acceleration polygons and instantaneous centers.

Modrey[3] applied kinematic coefficients (he referred to them as "influence coefficients") to the velocity and acceleration analysis of planar mechanisms in paradoxical configurations, configurations that did not amend themselves to standard methods of analysis. Benedict and Tesar[4] applied kinematic coefficients to the dynamic equation of motion of a machine.

In the early 1970s, when the digital computer was becoming a tool available to all engineers, making large amounts of computation commonplace, Hall introduced the vector loop method and kinematic coefficients into the undergraduate curriculum at Purdue University. There were no textbooks at that time which focused on the vector loop method, much like today. A few years prior to becoming Emeritus, Hall published a packet of course notes.[5] [6] Hall's "Notes . . ." was a dense introduction to the vector loop method and kinematic coefficients and their application to velocity and acceleration

[1]A.S. Hall Jr., *Mechanism and Machine Theory*, vol. 27, no. 3, p. 367, 1992
[2]R. Eksergian, *J. Franklin Inst.*, vol. 209, no. 1, January 1930 to vol. 201, no. 5, May 1931.
[3]J. Modrey, *ASME J. of Applied Mechanics*, vol. 81, pp. 184-188, 1957.
[4]C.E. Benedict and D. Tesar, *J. of Mechanisms*, vol. 6 pp. 383-403, 1971.
[5]A.S. Hall Jr., *Notes on Mechanism Analysis*, Balt Publishers, Lafayette, IN, 1981.
[6]A.S. Hall Jr., *Notes on Mechanism Analysis*, Waveland Press, Prospect Heights, IL, 1986.

analysis and to the inverse and forward dynamics problems. The forward dynamics problem and dynamic simulation of a machine became tenable through the use of kinematic coefficients. Hall can be credited with bringing this method into the undergraduate curriculum, disseminating this powerful tool to thousands of engineers.

This book is a major expansion of Hall's "Notes..." The majority of examples and problems are new here. Hall introduced the rolling contact equation for modeling rolling contacts in mechanisms. Here, the rolling contact equation is extended to include geartrains and transmissions so that rolling contacts in both mechanisms and geartrains are modeled by the same rolling contact equation. Kinematic coefficients have been extended here to describe limit positions, time ratio, dead positions, transmission angle, mechanical advantage and to develop Freudenstein's equation, arguably the most powerful synthesis tool available to a mechanism designer.

## Advantages of the Vector Loop Method

When studying the kinematics and dynamics of mechanisms and machines, the vector loop method and the kinematic coefficients offer several benefits:

- The method is purely algebraic. Geometric interpretations of kinematic properties such as limit positions and dead positions follow from interpretation of the algebraic expressions for the kinematic coefficients. Graphical methods and force analysis are not necessary to develop these properties of a mechanism. The algebraic results are also well suited to computerization. For example, the Jacobian matrix which is inverted to solve the position problem using Newton's method is the same Jacobian which needs to be inverted to solve for the kinematic coefficients. (Singularities of this Jacobian define the dead positions.) After solving the position problem there is no need to recompute the Jacobian since it is known from the last iteration in Newton's method. I believe young people today are more inclined to algebraic approaches rather than graphical approaches, and the vector loop method is purely algebraic.
- All kinematic and dynamic analysis is performed in a fixed frame of reference. In acceleration analysis the Coriolis terms arise naturally. The student no longer needs to recognize situations where the Coriolis terms must be included. Coriolis terms will appear during the differentiation required to find the second-order kinematic coefficients. All that is needed is the ability to apply the chain rule and the product rule for differentiation. Also, the equations that use kinematic coefficients to compute velocity and acceleration, either linear or rotational, have the same form.
- The most important feature of the vector loop method is that the dynamic simulation of a machine becomes a straightforward process. The same kinematic coefficients that describe the kinematic properties of velocity and acceleration appear seamlessly in the machine's equation of motion. This equation of motion is numerically integrated to yield the dynamic simulation. Development of the equation of motion, known as the Power equation, is systematic with the vector loop method.
- The Power equation also leads to a generic description of mechanical advantage as the ratio of the load force (or torque) to the driving force (or torque). The procedure

for finding an algebraic expression of a mechanism's mechanical advantage is systematic and does not involve free body diagrams. It also leads to another way to describe limit positions (infinite mechanical advantage) and dead position (zero mechanical advantage).

The vector loops are the basis of any mechanism's or machine's motion. They are the fundamental equations which completely define the machine's kinematics and dynamics. *The subject of mechanism and machine kinematics and dynamics is unified by the vector loop method* and for this reason I believe this book teaches the subject in the most condensed and straightforward manner.

I have found in teaching undergraduate courses on mechanisms and machines that most textbooks contain much more information than can be covered in one or even two semesters. Existing texts are excellent references but are too large to be appropriate for introductory courses. Of the nine chapters in this book, seven chapters (1-5 and 7-8) can be covered in a fifteen week semester.

## Objectives

There are two primary objectives of this book. The first objective is to teach students a modern method of mechanism and machine analysis, the vector loop method. The method predicts the kinematic performance of a mechanism, but most importantly, it makes the dynamic simulation of a machine tenable. This is through the machine's differential equation of motion, known as the Power equation. This nonlinear differential equation of motion is solved using a first-order Euler's method. The numerical solution allows a machine designer to select the motor or driver for a machine so that the machine achieves a desired time based response. This response may include things such as time to steady state operation or the flywheel size required to reduce steady state speed fluctuation to an acceptable level.

The second objective of the book is to give an introduction to mechanism synthesis. Of the myriad of synthesis problems that exist, we consider the two most basic, namely function generation and rigid-body guidance. In a single semester it is not possible to proceed further. Students interested in other types and more advanced forms of mechanism synthesis will need to proceed after the semester ends using the abundance of reference texts and archival literature that exists.

## Organization

The books begins with a basic introduction in **Chapter 1** which teaches students how to visually communicate the geometry of a mechanism or machine through skeleton diagrams and then determine the gross motion capabilities of the mechanism or machine. Vectors later associated to the vector loop method are used for this purpose. **Chapter 2** introduces position analysis through the vector loop method. Newton's method is taught to solve the nonlinear position equations. **Chapter 3** introduces rolling contacts. In addition to mechanisms, automotive transmissions, and an automotive differential are analyzed.

**Chapter 4** introduces kinematic coefficients. They are used for velocity and acceleration analysis. The physical meaning of the kinematic coefficients is given and they are used to algebraically predict the limit positions and dead positions of planar mechanisms. From these follow concepts such as time ratio and transmission angle.

**Chapter 5** reviews the inverse dynamics problem. Kinematic coefficients are used to compute the accelerations of mass centers and angular accelerations of links. There are three purposes for studying the inverse dynamics problem. The first is to show students how solving this problem is a tool for selecting the driver or motor for a machine. For this a numerical example of a motor selection is presented. The second purpose of chapter 5 is to introduce students to the three dimensional aspects in the force analysis of planar mechanisms and machines. The third purpose is to prepare students for **Chapter 6** where joint friction is studied. There students see that the linear inverse dynamics problem becomes nonlinear when friction is considered. The method of successive iterations is used to solve these nonlinear equations.

**Chapter 7** uses kinematic coefficients to develop the Power equation, the nonlinear differential equation of motion of a machine. This is the culmination of the vector loop method. A first-order Euler's method is taught to numerically integrate the equation, producing a dynamic simulation of the machine's motion, solving the forward dynamics problem. Continuing with the example in chapter 6, a dynamic simulation of the machine with its selected motor is developed. This verifies the motor selection and also shows students how a flywheel can be added to the machine to smooth its steady state operation. This example appears in both chapters 6 and 7 and illustrates the continuity between the inverse and forward dynamics problems. Kinematic coefficients and the Power equation are also used in chapter 7 to mathematically describe a mechanism's mechanical advantage.

**Chapter 8** begins discussion of mechanism synthesis. A vector loop is used to develop Freudenstein's equation and a systemic procedure for synthesizing four bar mechanisms which develop desired third-order kinematic coefficients is presented. This generates mechanisms that develop third-order Taylor's series approximations of the desired function generation.

**Chapter 9** presents the three position problem of a rigid body. This is the simplest possible guidance problem. It is beyond the scope of a fifteen week semester to consider the four and five position problem along with the issue of sequencing.

This course can be taught beginning with either analysis or synthesis, depending on the instructor's preference. Many mechanical engineering programs offer technical electives that cover mechanism synthesis in detail, in a semester long course. In that case the analysis portion of this book would be more important so as to prepare them for analyzing the designs they will develop in that follow on technical elective. Chapters 8 and 9 may then be neglected.

My personal preference is that synthesis is introduced first, in which case chapters 1, 8, and 9 are covered first. Chapters 8 and 9 are independent of the other chapters. The students naturally question how they might determine how well their designs are performing and how can they optimize amongst the many solutions they develop. This justifies the analysis that is covered by the remaining chapters, which should be taught in the order they are presented as chapter 3 depends on chapter 2, and so on.

## Problems and Exercises

This book contains three types of problems: Exercises, Programming Problems, and Design Problems. In light of the computational orientation of the vector loop method there are relatively few numerical results.

Exercises are intended as homework problems which develop a student's ability to model a mechanism and set up the correct equations for use in a computer program. Many examples and exercises develop flowcharts which show the logic of a computer program that uses the developed equations.

To keep the student grounded, there are Programming Problems that implement the modeling equations and generate numerical results and plots. These predict the mechanism's gross motion capabilities, force and torque transmitting capabilities, or time-based response of the machine.

Design problems are open-ended. A complete survey of possible solutions to these open-ended design problems also requires a computer program.

## Prerequisites

Students using this book should have taken the following courses, which are typically taken before or during the sophomore year:

1. Analytical Geometry
2. Calculus
3. Ordinary Differential Equations
4. Linear Algebra
5. Computer Programming (Matlab®, FORTRAN, C++, or any other scientific language)
6. Statics
7. Dynamics

## MindTap Online Course and Reader

In addition to the print version, this textbook is now also available online through MindTap, a personalized learning program. Students who purchase the MindTap version will have access to the books MindTap Reader and will be able to complete homework and assessment material online, through their desktop, laptop, or iPad. If your class is using a Learning Management System (such as Blackboard, Moodle, or Angel) for tracking course content, assignments, and grading, you can seamlessly access the MindTap suite of content and assessments for this course.

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# Acknowledgments

What a person knows or creates is a result of a life's experience, which is filled by input from other individuals, who in many cases were either teachers or role models. I would like to acknowledge three individuals without whom I would never have written this book. They, and those before them, have partial ownership of this book.

Professor Emeritus Allen Strickland Hall Jr. of the School of Mechanical Engineering at Purdue University was the finest of mentors and teachers. He was a patient man of few, but meaningful words. He was more of a guide than a teacher. Professor James A. Euler of the Mechanical Engineering Department of the University of Tennessee was a Ph.D. student at Purdue University when he led me at the age of nine, in the disassembly and rebuilding of a lawn mower engine. At that point I had disassembled many toys and small mechanical devices, or destroyed them as my parents said, generally failing to put them back together. This was the first machine I had encountered, and I began to wonder about the mechanical movements and their relative timing and coordination. Finally my father, Professor Emeritus Milomir Mirko Stanišić of the School of Aeronautics and Astronautics at Purdue University. He was an outstanding mathematical physicist and the finest of role models. He is continuously missed.

There are also many individuals who helped me in preparation of this book. Dr. Craig Goehler, of the Mechanical Engineering Department of Valparaiso University provided solutions to many of the homework problems. Ms. Mary E. Tribble reorganized many of the chapters and and corrected the errors of my Chicago-English. Mr. Conor Hawes checked the homework solutions. Dr. Zhuming Bi of Indiana University Purdue University Fort Wayne, Dr. Daejong Kim of University of Texas at Arlington, and Dr. D. Dane Quinn of The University of Akron, provided valuable input during the review process. Finally, I thank the Staff of Cengage Learning for their support and constructive criticisms.

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*University of Notre Dame*

CHAPTER 1

# Introduction

This text is an introduction to the kinematics and dynamics of mechanisms and machinery. Mechanisms are mechanical devices that consist of a system of interconnected rigid links. The purpose of a mechanism is to create a desired relative motion of the links. The dimensions of the links and the types of connections between the links dictate their relative motions.

Kinematics is the branch of mechanics that studies aspects of rigid body motions that are independent of both force and time. The subject matter is purely geometric. With regard to mechanisms, kinematics is the study of how the dimensions of links, and the connections between links, affect the relative motion of the links. There are two types of problems in mechanism kinematics. The first is the *kinematic analysis* problem. In this problem the mechanism is specified, and the relative motion of the links is determined. Chapters 2, 3, and 4 discuss kinematic analysis. In this text, the vector loop method is used for kinematic analysis. The second problem is the *kinematic synthesis* problem. In this problem the desired relative motion of the links is specified, and the mechanism that produces the desired motion is determined. Kinematic synthesis is a very broad field. The subject could not be fully examined in a one-semester course. Chapters 8 and 9 present two very basic methods of kinematic synthesis. The synthesis methods presented here are independent of the vector loop method and may be studied at any point. Both types of kinematics problems are equally important, and they go hand in hand. The synthesis problem typically yields many potential solutions. Analyzing the solutions determines the optimum design.

Dynamics is the branch of mechanics that studies the relationship between the forces and torques acting on a rigid body and its time-based motion. When a mechanism is used to transmit forces, torques, and power, it is referred to as a machine. With regard to machines, dynamics is the study of the relationship between the time-based motion of the machine's links and the forces and torques that are applied to the links. There are two types of dynamics problems. The first is the *inverse dynamics* problem. In the inverse dynamics problem the time-based motion of the machine is specified, and the driving forces or torques corresponding to this motion are determined. The second is the *forward dynamics* problem. In the forward dynamics problem the driving forces and torques are specified, and the resulting time-based motion of the machine is to be determined. These two problems also go hand in hand. For example, the inverse dynamics problem would be solved to size and select the motor needed to drive a machine under steady state conditions; then, using the selected motor, the forward dynamics problem would be solved to predict the time response of the machine.

In the remainder of this chapter we will look at the types of joints that exist between links in a planar mechanism. We will then learn how to make skeleton diagrams of planar mechanisms and how to compute the number of degrees of freedom (dof) in a planar mechanism.

## 1.1 JOINTS

We begin our study of mechanism kinematics by considering what types of connections, a.k.a. joints, exist between the links in a planar mechanism. The set of possible joints is very limited. First we need to make some definitions.

### DEFINITIONS:

**Link** *A rigid body.*

Links are numbered, and each is referred to by its number.

**Planar Motion** *A motion in which all points belonging to a link move in a plane known as the plane of motion, while simultaneously the link is free to rotate about an axis perpendicular to the plane of motion.*

Figure 1.1 shows an abstraction of two links. Link 1 is a fixed link. The small dark blue patch attached to 1 indicates that it is fixed. Link 2 is undergoing an unconstrained planar motion relative to 1. Let us consider this motion. Attach a frame of reference to 1, whose $X$-$Y$ axes define the plane of motion. We choose a reference point $Q$ and a reference line $\ell$ on 2.

We define the location of $Q$ relative to the origin of $X$-$Y$ with the vector $\bar{r}$, which has $X$ and $Y$ components $r_x$ and $r_y$,

$$\bar{r} = \begin{bmatrix} r_x \\ r_y \end{bmatrix}.$$

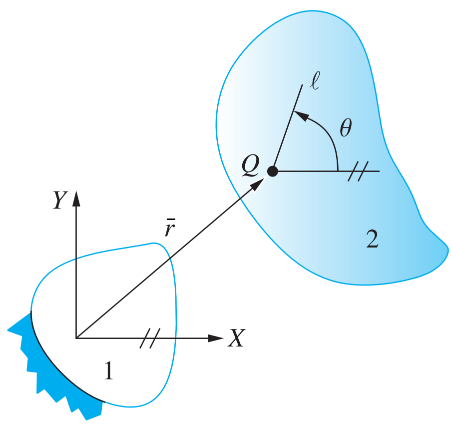


**FIGURE 1.1** A link undergoing planar motion

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We define the orientation of 2 relative to $X$-$Y$ with the angle $\theta$, measured from the positive direction of the $X$ axis to $\ell$. Since the motion of 2 relative to 1 is unconstrained, the two components of $\bar{r}$ and the angle $\theta$ are independent of each other. When $\bar{r}$ and $\theta$ are defined, the position of 2 relative to the $X$-$Y$ frame (i.e., link 1) is known. Thus we make the following statement:

An unconstrained link undergoing planar motion has three degrees of freedom. This is because three parameters—$r_x$, $r_y$, and $\theta$—must be specified in order to define the position of link 2. We also observe that without any joint between them, the motion of 2 relative to 1 has three degrees of freedom. In the upcoming discussion we will introduce various types of joints between links 1 and 2 and deduce how the joint has constrained their relative motion by eliminating degrees of freedom.

**DEFINITIONS:**

**Joint** *A permanent contact (connection) between two links.*

The words *joint*, *contact*, and *connection* can be used interchangeably. Although it may appear that a joint can come apart—for example, the joint between 3 and 4 in Figure 1.19—it is assumed that the contact is permanent and unbreakable.

**Joint Variable(s)** *The relative motion(s) between two connected bodies that occur(s) at the joint.*

**Independent Joint Variable(s)** *The joint variables associated with a particular type of joint that are independent, i.e., do not influence one another.*

There are two types of joints in planar mechanisms, $P_1$ joints and $P_2$ joints.

**$P_1$ Joint** *A joint between two links that constrains their relative motion by eliminating two degrees of freedom, thus allowing for one degree of freedom of relative motion.*

$P_1$ joints have one independent joint variable.

### 1.1.1 $P_1$ Joints

There are three types of $P_1$ joints: pin joints, sliding joints, and rolling joints.

#### *The Pin Joint*

The left side of Figure 1.2 shows 1links 1 and 2 with a pin joint between them. The right side shows the $X$-$Y$ frame attached to 1 and the point of reference on 2, point $Q$. We choose to place $Q$ at the center of the pin joint on 2, so $r_x$ and $r_y$ are fixed and these two degrees of freedom no longer exist. There is one remaining degree of freedom, $\theta$. Knowing $\theta$ defines the position of 2 relative to 1.

> Rotation $\theta$ is the independent joint variable of a pin joint.

The pin joint has eliminated two degrees of freedom from the motion of 2 relative to 1, allowing for a one degree of freedom motion of 2 relative to 1. This makes the pin joint a $P_1$ joint.

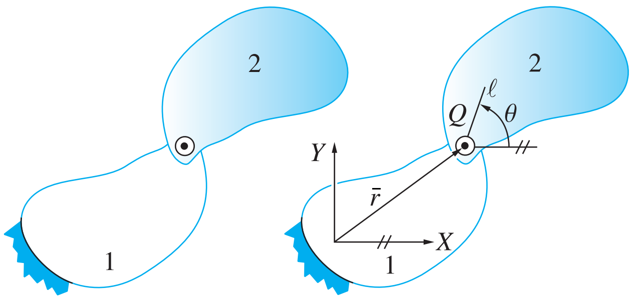


**FIGURE 1.2** Pin joint



### *The Multiple Pin Joint*

The left side of Figure 1.3 shows link 1, which is pinned to two links, 2 and 3. Clearly there are two pin joints there. Moving to the right in the figure, the two pin joints come closer so that distance *d* between them is reducing. There are still two pin joints. Finally, on the right hand side, *d* has gone to zero, so the two pin joints are coincident and appear to be one pin joint. However, that is not the case. There are still two pin joints there; they are just coincident. There is a simple rule that one can remember:

> The number of pin joints is one less than the number of links joined at that pin joint.

A mechanism is typically represented in a skeletal form, in what is known as a skeleton diagram. A skeleton diagram is a simplified drawing of a mechanism or machine that shows only the dimensions that affect its kinematics. We will look more into skeleton diagrams in Section 1.2. In a skeleton diagram, the three links pinned together on the right side of Figure 1.3 would have looked like the left side of Figure 1.4. The little black box indicates that link 1 is solid across the pin joint. The image on the right side

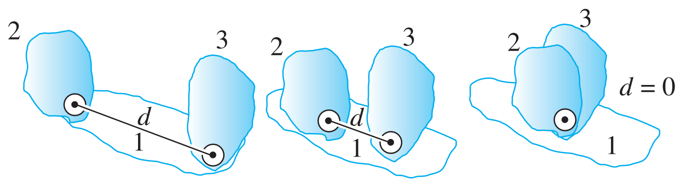


**FIGURE 1.3** Multiple pin joint



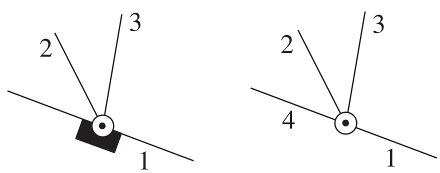


**FIGURE 1.4** Skeleton diagram representations of a multiple pin joint

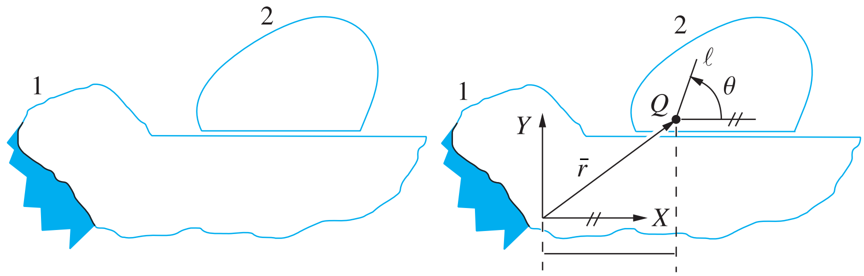


**FIGURE 1.5** Sliding joint



of Figure 1.4, which does not have the black box, represents four links joined together at the pin joint, so there would be three pin joints at that location.

### *The Sliding Joint*

Figure 1.5 shows links 1 and 2 with a sliding joint between them, the reference point $Q$, and the vector $\bar{r}$ and angle $\theta$ that define the position of 2 relative to 1. The values of $r_y$ and $\theta$ are constant, so these two degrees of freedom are eliminated. The remaining degree of freedom is $r_x$. Knowing $r_x$ defines the position of 2 relative to 1.

> Displacement $r_x$ is the independent joint variable of a sliding joint.

The sliding joint has eliminated two degrees of freedom from the motion of 2 relative to 1, allowing for a one-degree-of-freedom motion of 2 relative to 1. So the sliding joint is a $P_1$ joint.

Figure 1.6 shows four examples of how a sliding joint between 1 and 2 may be represented in a skeleton diagram. The dimensions of the slider block, 2, are unimportant and arbitrary. The only significant feature of a sliding joint is the direction of sliding. The top left representation is most common. In some skeleton diagrams the slider block will be drawn as being encased in the mating body, as shown on the top right. In this representation you might think there are two parallel sliding joints, one at the top and the other at the bottom of 2. You should consider this to be only one sliding contact, because the second parallel sliding contact is redundant. If the two sliding contacts in

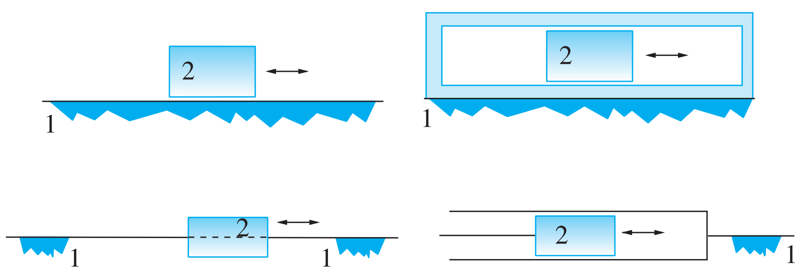


**FIGURE 1.6** Skeleton diagram representation of a sliding joint

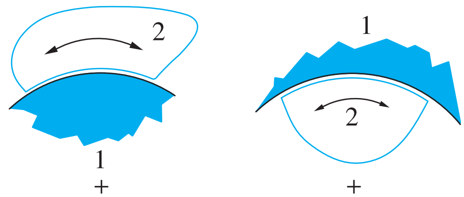


**FIGURE 1.7** Circular sliding joints

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that image were not parallel, then they would each count as a sliding contact. The bottom left depicts a block sliding along a shaft. The bottom right depicts a piston sliding in a cylinder, such as you see in the skeleton diagram of the front loader in Figure 1.16.

### *The Circular Slider*

Figure 1.7 depicts circular sliding joints. Circular sliders in fact are oversized pin joints. The "+" indicates the center of the pin joint. As in a pin joint, the rotation indicated in Figure 1.7 is the joint variable of a circular slider. Like the revolute joint and straight sliding joint, the circular slider allows one degree of freedom in the motion of 2 relative to 1, eliminates two degrees of freedom, and is a $P_1$ joint.

Pin joints and sliding joints are line contacts. The bodies connected by a pin joint in Figures 1.2, 1.3, 1.4, and 1.7 contact along a line that is a circle (or part of a circle in 1.7). The bodies connected by a sliding joint in Figures 1.5, 1.6, and 1.7 also contact along a line that is straight in 1.5 and 1.6 and part of a circle in 1.7. As you will see, the remaining joints are all point contacts. This makes the pin joint and the sliding joint readily identifiable as $P_1$ joints, and we can state a simple rule:

> Any joint that involves a line contact between the connected bodies is a $P_1$ joint.

### *The Rolling Joint*

Figure 1.8 shows a rolling joint between links 1 and 2. It is simplest to consider the case when 1 is flat and 2 is circular, although our discussion applies to a rolling joint between two links of any shape. Assign the reference point $Q$ at the center of 2. The figure shows the vector $\bar{r}$ and angle $\theta$ that define the position of 2 relative to 1.

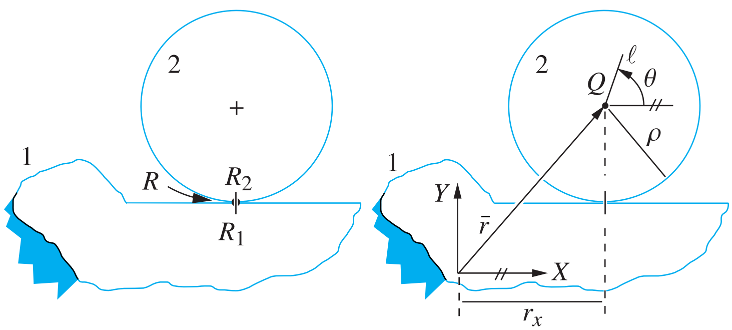


**FIGURE 1.8** Rolling joint

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The two bodies contact at point $R$, shown on the left. At point $R$ there exist a pair of instantaneously coincident points $R_1$ and $R_2$ belonging to links 1 and 2, respectively. A hash mark is drawn at the point of contact to indicate that this is a rolling joint. The hash mark is drawn in the direction of the common normal to the two bodies at the point of contact. Point $R$ and the hash mark are the location of a no-slip condition between the two rolling links. Let $\bar{v}_{r_1}$ and $\bar{v}_{r_2}$ represent the velocity of points $R_1$ and $R_2$ respectively. The no-slip condition means that the relative velocity between $R_1$ and $R_2$ is zero, i.e., $\bar{v}_{r_1/r_2} = \bar{0}$ and thus $\bar{v}_{r_1} = \bar{v}_{r_2}$. The no-slip condition also means there is no rubbing between the two bodies at the point of contact.

In the rolling joint, $r_y$ is constant while $r_x$ and $\theta$ are changing. $r_x$ and $\theta$ are the joint variables of a rolling joint, but they are not independent. Their changes, $\Delta r_x$ and $\Delta\theta$, are related to each other by the no-slip condition, namely

$$\Delta r_x = -\rho\Delta\theta. \tag{1.1}$$

The negative sign appears because a positive (counterclockwise) $\Delta\theta$ causes $Q$ to roll to the left, and this corresponds to a $-\Delta r_x$. In this equation either $\Delta\theta$ or $\Delta r_x$ is independent, but not both. So there is only one degree of freedom.

Rotation $\theta$, *or* displacement $r_x$, is the independent joint variable of a rolling joint.

We conclude that the rolling joint allows one degree of freedom for the motion of 2 relative to 1 and eliminates two degrees of freedom. Thus the rolling joint is a $P_1$ joint. For purposes of graphical communication, *a rolling joint in a skeleton diagram will always be indicated by a hash mark at the point of contact*, as shown in the figure. In many instances both links in a rolling joint are circular, as shown in Figure 1.9. Rolling joints are typically achieved by gears, and the two circular shapes in Figure 1.9 are the *pitch circles* of those gears.

We have considered all the known $P_1$ joints—the pin joint, the sliding joint, and the rolling joint—and we have seen that $P_1$ joints allow for one degree of freedom in the relative motion of the two connected links. Thus the following statement is true.

All $P_1$ joints eliminate two degrees of freedom.

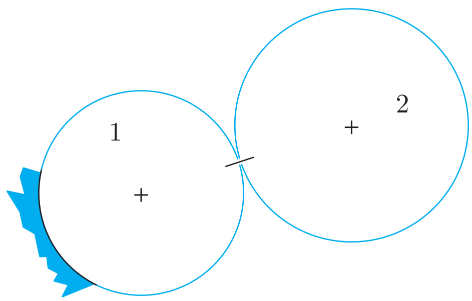


**FIGURE 1.9** Rolling joint between circular shapes

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## 1.1.2 $P_2$ Joints

**DEFINITION:**

**$P_2$ Joint** *A joint between two links that constrains their relative motion by eliminating one degree of freedom, thus allowing for two degrees of freedom of relative motion.*

$P_2$ joints have two independent joint variables. The only known $P_2$ joint is the slipping joint.

### *The Slipping Joint*

Figure 1.10 shows a slipping joint between links 1 and 2. In a slipping joint, links 1 and 2 contact at a point, as in the rolling joint. Unlike the rolling joint, in a slipping joint there is rubbing at the point of contact and the no-slip condition, Equation (1.1) does not apply. In a skeleton diagram the only distinction between a rolling joint and a slipping joint is the hash mark. The hash mark, as in Figure 1.8, is used to indicate that there *is* a no-slip condition at the point of contact, and this corresponds to a rolling joint. *There is no hash mark at the point of contact in a slipping joint.* Without Equation (1.1), $r_x$ and $\theta$ are independent of each other, so the slipping contact allows two degrees of freedom, and eliminates one degree of freedom, in the motion of 2 relative to 1.

> Rotation $\theta$ *and* displacement $r_x$ are the independent joint variables of a slipping joint.

The slipping joint is a $P_2$ joint, and the following statement is true of all $P_2$ joints.

> A $P_2$ joint eliminates one degree of freedom.

Figure 1.11 shows several possible depictions of a slipping joint in a skeleton diagram. The version of the slipping joint shown at the bottom right of Figure 1.11 is referred to as a "pin in a slot" joint. In this case, when the two surfaces of 1 that contact the circular portion of 2 are parallel, there is only one $P_2$ joint and not two. The second $P_2$ joint would be redundant. If the two surfaces of 1 that contact 2 were not parallel, they would count as two $P_2$ joints.

To visually communicate the geometry and structure of a mechanism that are important to its kinematics, we use *skeleton diagrams*. The next section introduces you

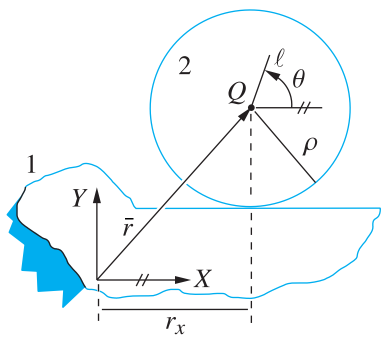


**FIGURE 1.10** Slipping joint

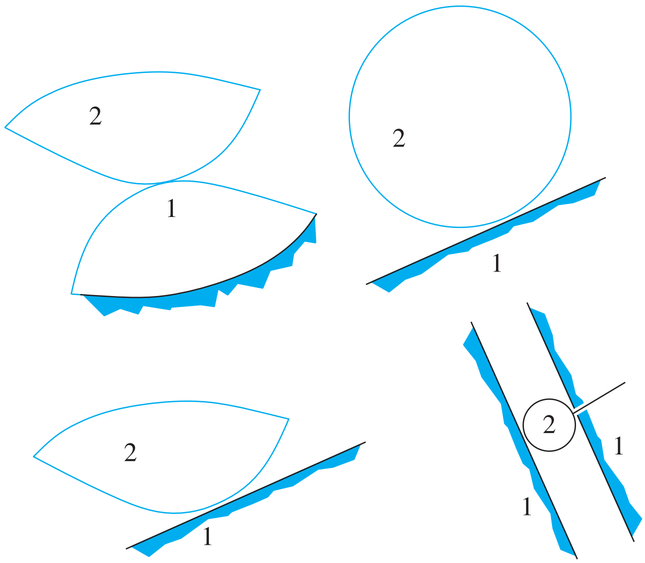


**FIGURE 1.11** Skeleton diagram representations of slipping joints

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to skeleton diagrams and also shows you several examples of mechanisms and machines that we see around us in our everyday lives.

## 1.2 SKELETON DIAGRAMS

A skeleton diagram is a simplified drawing of a mechanism or machine that shows only the dimensions that affect its kinematics. Figure 1.12 shows a connecting rod with its attached piston, from an internal combustion engine. The connecting rod and piston both have many geometric features, mostly associated with issues of strength and the size of the bearing at each joint. These features are kinematically unimportant.

The *only* geometric feature of either the connecting rod or the piston that is important to the kinematics of the mechanism containing them is the distance, $l$, between the

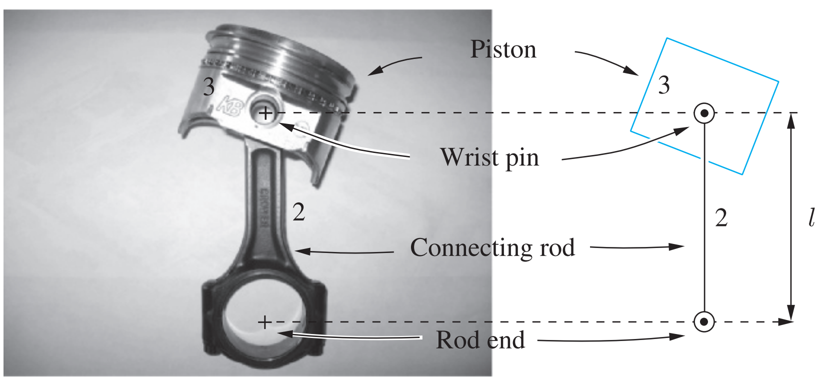


**FIGURE 1.12** A connecting rod and its skeletal representation

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